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# "LMI approach to mixed performance objective controllers: application to Robust $H_2$ Synthesis"

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# LMI approach to mixed performance objective controllers: application to Robust $\mathcal{H}_2$ Synthesis

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**Abstract**

The problem of synthesizing a controller for plants subject to arbitrary, finite energy disturbances and white noise disturbances via Linear Matrix Inequalities (LMIs) is presented. This is achieved by considering white noise disturbances as belonging to a constrained set in $l_2$. In the case of where only white noise disturbances are present, the procedure reduces to standard $\mathcal{H}_2$ synthesis. When arbitrary, finite energy disturbances are also present, the procedure may be used to synthesize general mixed performance objective controllers, and for certain cases, Robust $\mathcal{H}_2$ controllers.

## 1 Introduction

In the standard robust control paradigm, the signal space which characterizes performance is equivalent to that which captures a system's uncertainty. For example, $\mathcal{H}_\infty$ tools are used when dealing with bounded energy (or power) gain uncertainty (see [12], [10], [18]), while when working with $l_\infty$ disturbances, the uncertainty is assumed to be of finite amplitude gain (see [9]). While it is often the case that the particular characterization of the uncertainty is not critical to the design process, the signal space used to characterize the performance often is. In particular, one of the common complaints among control design engineers which use $\mathcal{H}_\infty$ methods is that the resulting designs tend to be sluggish and overly conservative. As an alternative, $\mathcal{H}_2$ designs are often employed, but they lack the robustness properties of $\mathcal{H}_\infty$ designs (see [5]) which can readily be extended to encompass a system's uncertainty. The attractive feature of $\mathcal{H}_2$ designs is their gain interpretation; they minimize the power output when the disturbances are assumed to be white noise or impulses. This is in contrast to $\mathcal{H}_\infty$ designs, which minimize the energy to energy (or power to power) gain; in many applications, modeling the disturbances as arbitrary signals is a poor modeling choice, and thus $\mathcal{H}_\infty$ designs lead to low performance controllers.

A desirable control design strategy would be one which has the input-output gain interpretation of the $\mathcal{H}_2$ norm, but can readily accommodate $\mathcal{H}_\infty$ bounds on the uncertainty.

In [15], a framework is developed whereby white noise signals are captured in a deterministic setting. The main motivation behind this approach was the reconciling of the worst case setting, natural when considering robustness issues, with the stochastic setting. This framework proved very natural when addressing the so-called *Robust $\mathcal{H}_2$ Analysis* problem, which was solved in [13] and [14].

This approach will be used in this paper to tackle the problem of *Robust $\mathcal{H}_2$ Synthesis* for a restricted class of problems; in particular, rank one synthesis problems with time varying uncertainty. This will be achieved by solving an auxiliary problem, that of synthesizing a controller for plants subject to arbitrary, finite energy disturbances and white noise disturbances.

Other work in this area includes [2], [6], [19] and [17]. Also worth noting is research on the so called *mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$* problem, where only nominal stability and nominal $\mathcal{H}_2$ performance are considered (see [1], [8]).

The paper is organized as follows: we begin with some mathematical preliminaries, followed by a review of the notions introduced in [15] with regards to capturing noise signals as elements of a set. The problem is then posed and solved, followed by a discussion on computational issues. The problem of robust disturbance rejection is then addressed, and it is shown how these types of problems can be re-formulated in the general problem setup previously solved. We conclude with an example which illustrates the tools developed and their numerical properties.

## 2 Preliminaries

Most of the notation in this paper is standard. We restrict ourselves to discrete time systems. The space of square summable sequences is denoted $l_2$; when the spatial structure is relevant, it is referred to as $l_2^p$. The 2-norm of a signal $d$ in $l_2$ is denoted $\|d\|$. The unit ball of $l_2$ signals is denoted $\mathbf{B}l_2$, and consists of all $l_2$ signals whose norm is less than or equal to one. The discrete time, unit delay operator is denoted $\lambda$. The truncation operator $P_T$ is defined as

$$(P_T x)(t) := \begin{cases} x(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases} \tag{1}$$

Causal, finite dimensional, linear time invariant systems will be denoted FDLTI. A causal linear map $G$ over $l_2$ is bounded if the restriction of $G$ to $l_2$ is a bounded operator, with induced $l_2$ norm denoted $\|G\|$. The transfer function representation of FDLTI system $G$

is denoted $\hat{G}(\lambda)$. The linear fractional transformation (LFT) between two systems $G$ and $K$ is denoted $G \star K$, and is defined as:

$$G \star K := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{2}$$

where

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

when the inverse of $(I - G_{22}K)$ is well defined. $\mathbf{B}\Delta$ is the set of linear, but otherwise arbitrary, operators whose induced 2-norm is less than or equal to 1. For two subsets of $l_2$, $S_1$ and $S_2$, the *approximation error* of $S_1$ with $S_2$ is defined as

$$D(S_1, S_2) := \sup_{s_1 \in S_1} \inf_{s_2 \in S_2} \|s_1 - s_2\| \tag{3}$$

$D(S_1, S_2)$ is a measure of how well $S_2$ captures the elements of $S_1$. For a constant matrix $A$, its maximal element is denoted

$$\|A\|_{\mathrm{M}} := \max_{i,j} |A_{i,j}| \tag{4}$$

## 3 Deterministic Noise Sets

We begin by reviewing the notions introduced in [15] to capture white noise in sets. Given a signal $n \in l_2^m$, its *autocorrelation function* is defined as

$$R_n(\tau) := \sum_{t=-\infty}^{\infty} n(t)n^{\mathrm{T}}(t + \tau) \tag{5}$$

Note that there is no time averaging in the definition above, as would be used for power signals, since we are dealing with finite energy signals. Given positive integer $N$ and positive number $\gamma$, we define the following set of autocorrelation functions:

$$\mathcal{R}^m_{N,\gamma} := \left\{ R(\tau) : \mathbb{Z} \to \mathbb{R}^{m \times m} \;\middle|\; \begin{array}{l} R(-\tau) = R^{\mathrm{T}}(\tau) \\ \|R(0) - I\|_{\mathrm{M}} \leq \gamma \\ \|R(\tau)\|_{\mathrm{M}} \leq \gamma, \;\; 1 \leq \tau \leq N \end{array} \right\} \tag{6}$$

and corresponding signal set

$$\mathcal{W}^m_{N,\gamma} := \left\{ n \in l_2^m | R_n(\tau) \in \mathcal{R}^m_{N,\gamma} \right\} \tag{7}$$

In particular, when $\gamma = 0$, we have

$$\mathcal{W}^m_N := \mathcal{W}^m_{N,0} \tag{8}$$

It is shown in [15] how the worst case gain from $\mathcal{W}^m_N$ to $l_2$ approaches the $\mathcal{H}_2$ norm in the limit as $N$ goes to infinity, the rate of convergence being exponential.

## 4 Problem Formulation

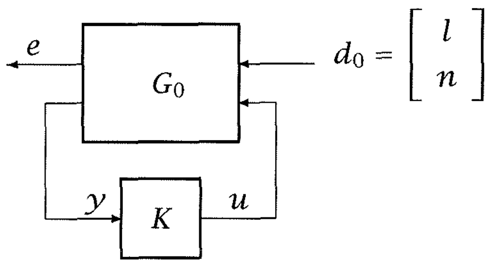


Figure 1: Problem Formulation

The problem formulation is as follows; given positive integer $N$ and FDLTI system $G_0$

$$G_0 = \begin{bmatrix} G_{11}^l & G_{11}^n & G_{12} \\ G_{21}^l & G_{21}^n & G_{22} \end{bmatrix} \tag{9}$$

find internally stabilizing (see [21]) FDLTI system $K$ such that

$$\sup_{l \in \mathbf{Bl}_2, n \in \mathcal{W}_N^m} \|(G_0 \star K) d_0\|^2 < 1 \tag{10}$$

The performance objective is absorbed into plant $G_0$; in general, the performance objective will be decreased until a controller no longer exists which satisfies (10). We will later show how the solution to the above problem may be used to solve a variety of robustness problems. The above problem formulation is interesting in its own right, however; it may be the case that some of the disturbance signals may be arbitrary, in this case $l$, while the others are white noise, such as $n$. For example, this may be the case when tracking a reference signal $l$ (which may be weighted to restrict tracking over a certain frequency range), in the presence of sensor noise or other random disturbance $n$.

## 5 Solution

We will provide a solution to the above problem in a series of steps. The first consists of recalling the solution of controller synthesis when the disturbances are subject to implicit constraints, presented in [3]. The next step consists of parametrizing sets $\mathcal{W}_N^m$ in image form. The final step is to combine these image representations with the solution provided in the first step to solve (10).

### 5.1 Synthesis with implicit constraints

The following is a review of the main results in [3]. Consider the following subset of $l_2$,

$$\mathcal{H} = \left\{ d \in l_2 \;\middle|\; \begin{array}{ll} \|H_i d\|^2 \leq 1 & 1 \leq i \leq c \\ \|L_j d\|^2 = \|J_j d\|^2 & 1 \leq j \leq \bar{c} \end{array} \right\} \tag{11}$$

where the $H_i$, $L_j$ and $J_j$ are constant matrices. Consider the following constrained feasibility problem: let system $G$ and set $\mathcal{H}$ be given. Find a stabilizing controller $K$ such that

$$\sup_{d \in \mathcal{H}} \|(G \star K)d\|^2 < 1 \tag{12}$$

It is shown in [3] that such a controller $K$ exists if and only if there exist symmetric matrices $\mathbf{S}, \mathbf{T}, \mathbf{W}, \mathbf{Z}$ (of a given spatial structure) such that

$$V \left( R^T \begin{bmatrix} \mathbf{S} & 0 \\ 0 & I \end{bmatrix} R - \begin{bmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{Z} \end{bmatrix} \right) V^T < 0 \tag{13}$$

$$U^T \left( R \begin{bmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{W} \end{bmatrix} R^T - \begin{bmatrix} \mathbf{T} & 0 \\ 0 & I \end{bmatrix} \right) U < 0$$

$$\begin{bmatrix} \mathbf{T} & I \\ I & \mathbf{S} \end{bmatrix} > 0$$

$$\begin{bmatrix} \mathbf{Z} & I \\ I & \mathbf{W} \end{bmatrix} > 0$$

$$C^T \mathbf{Z} C < 1$$

where $U$, $V$, $R$ and $C$ are constant matrices which depend on the state space representation for $G$ and the constraint set $\mathcal{H}$. A state space representation for $K$ may then be constructed from $R$, $\mathbf{S}$ and $\mathbf{T}$ (the details may be found in [3] and [11]). The above is a convex feasibility problem, and may be solved using numerical packages such as *LMI Lab* [7].

### 5.2 Image Representation for $\mathcal{W}_N^m$

The solution provided in the previous section cannot be utilized to directly solve (10); in order to directly specify $\mathcal{W}_N^m$, $H_i$, $L_j$ and $J_j$ need to be systems, not constant matrices (see [2]). What we would like to do is construct an alternate representation for $\mathcal{W}_N^m$ which is consistent with the solution provided in the previous section.

Let $N$ and $m$ be given. Define

$$\begin{aligned} \hat{U}_k(\lambda) &:= \frac{1+\lambda^k}{\sqrt{8}N} \\ \hat{\bar{U}}_k(\lambda) &:= \frac{1-\lambda^k}{\sqrt{8}N} \qquad\qquad 1 \le k \le N \\ U &:= [U_1 \ \bar{U}_1 \ \ldots \ U_N \ \bar{U}_N] \\ V &:= \text{diag } [U, \ U, \ \ldots, \ U] \end{aligned} \tag{14}$$

where $V \in \mathcal{RH}_\infty^{m\times(2mN)}$, ie., $V$ consists of $m$ copies of $U$ along the "diagonal". Then it may be verified that

$$\hat{V}\hat{V}^* = \frac{1}{2N} I_m \tag{15}$$

ie., $\hat{V}$ and $\hat{U}$ are co-inner, and $\left\|\hat{V}\right\|_\infty = \left\|\hat{U}\right\|_\infty = \frac{1}{\sqrt{2N}}$.

Define $\tilde{n} \in l_2^{2mN}$ as follows:

$$\begin{aligned} \tilde{n} &:= (\tilde{n}_1, \ \ldots, \ \tilde{n}_m) \\ \tilde{n}_i &:= (n_{i,1}, \ \bar{n}_{i,1}, \ \ldots, \ n_{i,N}, \ \bar{n}_{i,N}) \end{aligned} \tag{16}$$

and the following set of constraints:

| | | | |
|---|---|---|---|
| $\mathbf{C}_1$ : | $\langle n_{i,k}, n_{i,k}\rangle \le 1, \ \langle \bar{n}_{i,k}, \bar{n}_{i,k}\rangle \le 1$ | $1 \le i \le m,$ | $1 \le k \le N$ |
| $\mathbf{C}_2$ : | $\langle n_{i,k}, n_{j,k}\rangle - \langle \bar{n}_{i,k}, \bar{n}_{j,k}\rangle = 0$ | $1 \le i < j \le m,$ | $1 \le k \le N$ |
| $\mathbf{C}_3$ : | $\langle n_{i,k}, \bar{n}_{j,k}\rangle - \langle \bar{n}_{i,k}, n_{j,k}\rangle = 0$ | $1 \le i < j \le m,$ | $1 \le k \le N$ |
| $\mathbf{C}_4$ : | $\langle n_{i,k}, n_{j,k}\rangle + \langle \bar{n}_{i,k}, \bar{n}_{j,k}\rangle = 0$ | $1 \le i < j \le m,$ | $k = 1$ |

The constraint set $\mathcal{N}$ is then defined as:

$$\mathcal{N} := \{\tilde{n} \in l_2^{2mN} | \mathbf{C}_1, \ \mathbf{C}_2, \ \mathbf{C}_3, \ \mathbf{C}_4 \text{ are satisfied}\} \tag{17}$$

The image set $\widetilde{\mathcal{W}}_{N,\gamma}^m$ may now be defined:

$$\widetilde{\mathcal{W}}_{N,\gamma}^m := \{n \in l_2^m | n = V\tilde{n}, \tilde{n} \in \mathcal{N}, \|n_i\| \ge 1-\gamma \text{ for } 1 \le i \le m\} \tag{18}$$

For $\gamma = 1$, we define

$$\widetilde{\mathcal{W}}_N^m := \widetilde{\mathcal{W}}_{N,1}^m \tag{19}$$

which corresponds to no explicit norm constraint on $n_i$. The following theorem outlines how $\mathcal{W}_N^m$ and $\widetilde{\mathcal{W}}_{N,\gamma}^m$ are related, and is crucial to the synthesis results which follow:

**Theorem 1**

1. $\widetilde{\mathcal{W}}^m_{N,0} = \mathcal{W}^m_N$.

2. $\mathcal{W}^m_N \subset \widetilde{\mathcal{W}}^m_{N,\gamma}$ *for* $\gamma \geq 0$.

3. $D(\widetilde{\mathcal{W}}^m_{N,\gamma}, \mathcal{W}^m_N)$ *is upper semi-continuous as a function of* $\gamma$ *at* $\gamma = 0$.

Before proving Theorem 1, we will need the following two Lemmas:

**Lemma 1**

$$S := \{n \in l_2^m | 2NV^*n \in \mathcal{N}, \|n_i\|^2 = 1 \text{ for } 1 \leq i \leq m\} = \widetilde{\mathcal{W}}^m_{N,0} \tag{20}$$

**Proof of Lemma 1:**

It is clear by setting $\tilde{n} = 2NV^*n$ that $S \subset \widetilde{\mathcal{W}}^m_{N,0}$; we will thus show that $\widetilde{\mathcal{W}}^m_{N,0} \subset S$. Let $n \in \widetilde{\mathcal{W}}^m_{N,0}$, with corresponding $\tilde{n}$. For each component, by constraints $\mathbf{C}_1$, $\|\tilde{n}_i\|^2 \leq 2N$, which implies $\|n_i\|^2 \leq 1$. Thus $\|n_i\|^2 = 1$, $\|\tilde{n}_i\|^2 = 2N$.

Since $\hat{U}$ is co-inner, $\exists \hat{U}_\perp \in \mathcal{RH}_\infty^{(2N-1)\times 2N}$ such that $\sqrt{2N}\begin{bmatrix} \hat{U} \\ \hat{U}_\perp \end{bmatrix}$ is unitary. Thus for each $i$, $\tilde{n}_i$ can be uniquely decomposed as

$$\tilde{n}_i = U^*v_i + U_\perp^* w_i \tag{21}$$

where $v_i \in l_2$, $w_i \in l_2^{2N-1}$. Furthermore, $\|\tilde{n}_i\|^2 = \|U^*v_i\|^2 + \|U_\perp^* w_i\|^2$. From this we may conclude that $v_i = 2Nn_i$, and that $\|v_i\| = \|U^*v_i\|^2 = \|\tilde{n}_i\|^2 = 2N$; thus $w_i = 0$, and $\tilde{n}_i = 2NU^*n_i$. This gives $\tilde{n} = 2NV^*n$, as required.

□

**Lemma 2** *Given* $R \in \mathcal{R}^m_{N,\gamma}$, *where* $\gamma < \frac{1}{m(N+1)^2}$, *there exists a signal* $x \in l_2^+$ *such that* $R_x(\tau) = R(\tau)$ *for* $\tau \in [-N, N]$.

The proof of Lemma 2 may be found in the Appendix. We are now in a position to prove Theorem 1:

**Proof of Theorem 1:**

We begin by showing how $n$ is constrained when $2NV^*n \in \mathcal{N}$. Let $\tilde{n} = 2NV^*n$. Thus

$$\begin{aligned} n_{i,k} &= 2NU_k^* n_i \\ \tilde{n}_{i,k} &= 2N\bar{U}_k^* n_i \qquad \text{for } 1 \leq i \leq m,\ 1 \leq k \leq N \end{aligned} \tag{22}$$

For the above, it can be verified that constraints $\mathbf{C}_1$ to $\mathbf{C}_4$ are equivalent to :

$$\begin{array}{llll} \mathbf{C}_1: & \|n_i\|^2 + \langle n_i, \lambda^k n_i\rangle \leq 1 & 1 \leq i \leq m, & 1 \leq k \leq N \\ & \|n_i\|^2 - \langle n_i, \lambda^k n_i\rangle \leq 1 & 1 \leq i \leq m, & 1 \leq k \leq N \\ \mathbf{C}_2: & \langle n_i, \lambda^k n_j\rangle + \langle n_i, \lambda^{-k} n_j\rangle = 0 & 1 \leq i < j \leq m, & 1 \leq k \leq N \\ \mathbf{C}_3: & \langle n_i, \lambda^k n_j\rangle - \langle n_i, \lambda^{-k} n_j\rangle = 0 & 1 \leq i < j \leq m, & 1 \leq k \leq N \\ \mathbf{C}_4: & \langle n_i, n_j\rangle = 0 & 1 \leq i < j \leq m & \end{array}$$

**Proof of 1:** Let $n \in \mathcal{W}_N^m$. The above constraints are then trivially satisfied, proving $n \in S$; thus by Lemma 1, $n \in \widetilde{\mathcal{W}}_{N,0}^m$. Now let $n \in \widetilde{\mathcal{W}}_{N,0}^m$, or equivalently, $n \in S$. It is then straightforward to show that the above constraints imply that $n \in \mathcal{W}_N^m$, as required.

**Proof of 2:** This follows from 1. and $\widetilde{\mathcal{W}}_{N,y}^m \subset \widetilde{\mathcal{W}}_{N,y_0}^m$ for $0 \le y \le y_0$.

**Proof of 3:** Let $\epsilon > 0$ be given. It will be shown that there exists a $y_0 > 0$ such that for all $0 \le y \le y_0$, $D(\widetilde{\mathcal{W}}_{N,y}^m, \mathcal{W}_N^m) < \epsilon$. Let $y_0 > 0$ be fixed. For any $0 \le y \le y_0$, $n \in \widetilde{\mathcal{W}}_{N,y}^m$ and corresponding $\tilde{n}$, we may decompose $\tilde{n}$ as in equation (21) yielding:

$$\tilde{n}_i = 2NU^* n_i + U_\perp^* w_i, \quad 1 \le i \le m \tag{23}$$

Since $\|\tilde{n}_i\| \le \sqrt{2N}$ and $\|n_i\| \ge 1 - y_0$, it follows that $\|w_i\| \le 2N\sqrt{2y_0}$. Applying constraints $\mathbf{C_1}$ through $\mathbf{C_4}$ to (23) results in

$$\begin{array}{llll}
\mathbf{C_1}: & \|n_i\|^2 + \langle n_i, \lambda^k n_i \rangle \le 1 + O(\|w_i\|); & 1 \le i \le m, & 1 \le k \le N \\
 & \|n_i\|^2 - \langle n_i, \lambda^k n_i \rangle \le 1 + O(\|w_i\|) & 1 \le i \le m, & 1 \le k \le N \\
\mathbf{C_2}: & |\langle n_i, \lambda^k n_j \rangle + \langle n_i, \lambda^{-k} n_j \rangle| \le O(\|w_i\|) & 1 \le i < j \le m, & 1 \le k \le N \\
\mathbf{C_3}: & |\langle n_i, \lambda^k n_j \rangle - \langle n_i, \lambda^{-k} n_j \rangle| \le O(\|w_i\|) & 1 \le i < j \le m, & 1 \le k \le N \\
\mathbf{C_4}: & |\langle n_i, n_j \rangle| \le O(\|w_i\|) & 1 \le i < j \le m &
\end{array}$$

We may thus conclude that there exists a constant $C$, independent of $y_0$, such that $n \in \mathcal{W}_{N,C\sqrt{y_0}}$. Since $n \in l_2$, there exists a $T \in \mathbb{Z}^+$ such that $\|n_i - \frac{P_T n_i}{\|P_T n_i\|}\| \le 2y_0$. Define

$$\begin{aligned}
\hat{n} &:= (\hat{n}_1, \ldots, \hat{n}_m) \\
\hat{n}_i &:= (1 - \frac{\epsilon}{6\sqrt{m}}) \frac{P_T n_i}{\|P_T n_i\|}
\end{aligned} \tag{24}$$

It can be shown that for $y_0 \le \frac{1}{2C}$, $\hat{n} \in \mathcal{W}_{N,2C\sqrt{y_0}}$. Furthermore, $\|n - \hat{n}\| \le \frac{\epsilon}{6} + 2\sqrt{m}y_0$. Let

$$\begin{aligned}
C_1 &= 1 - \left(1 - \frac{\epsilon}{6\sqrt{m}}\right)^2 \\
R(\tau) &= -C_1^{-1} R_{\hat{n}}(\tau), \qquad 1 \le |\tau| \le N \\
R(0) &= C_1^{-1} (I - R_{\hat{n}}(0))
\end{aligned} \tag{25}$$

Then it can be verified that $R \in \mathcal{R}_{N,2C\sqrt{y_0}C_1^{-1}}^m$. By Lemma 2, for $y_0 \le \frac{C_1^2}{8C^2 m^2 (N+1)^4}$, there exists signal $x \in l_2^+$ such that $R_x(\tau) = R(\tau)$ for $\tau \in [-N, N]$.
Define

$$d = \hat{n} + C_1 \lambda^{N+T+1} x \tag{26}$$

Then

$$\begin{aligned}
R_d(\tau) &= R_{\hat{n}}(\tau) + C_1 R_x(\tau) \\
&= 0 \qquad \tau \ne 0 \\
&= I \qquad \tau = 0
\end{aligned} \tag{27}$$

Thus $d \in \mathcal{W}_N^m$. Furthermore,

$$\begin{aligned} \|n - d\| &\leq \|n - \hat{n}\| + \|\hat{n} - d\| \\ &\leq \frac{\epsilon}{6} + 2\sqrt{m}\gamma_0 + \frac{\epsilon}{3} \end{aligned} \tag{28}$$

Choosing

$$\gamma_0 = \min\left\{\frac{1}{2C}, \frac{C_1^2}{8C^2m^2(N+1)^4}, \frac{\epsilon}{6\sqrt{m}}\right\} \tag{29}$$

gives the required results.

$\square$

### 5.3 Image Representations and Implicit Constraints

We are now in a position to provide a solution to (10) by combining the results of Sections 5.1 and 5.2. The following theorem outlines how we may replace set $\mathcal{W}_N^m$ with $\widetilde{\mathcal{W}}_N^m$. The idea is to penalize disturbance $n \in \widetilde{\mathcal{W}}_N^m$ in such a way that the worst case error will occur when $n$ is as large as possible; this in effect will force $n$ to be in $\mathcal{W}_N^m$:

**Theorem 2** *Given $M_l$ and $M_n \in \mathcal{RH}_\infty$, then*

$$\sup_{l \in \mathbf{Bl}_2, n \in \mathcal{W}_N^m} \|M_l l + M_n n\|^2 < 1 \tag{30}$$

*if and only if there exists $k_0$ such that $\forall k \geq k_0$,*

$$\sup_{l \in \mathbf{Bl}_2, n \in \widetilde{\mathcal{W}}_N^m} \left\| \begin{matrix} kn \\ M_l l + M_n n \end{matrix} \right\|^2 < mk^2 + 1 \tag{31}$$

**Proof:**
**(31) $\Longrightarrow$ (30):** Since $\mathcal{W}_N^m \subset \widetilde{\mathcal{W}}_N^m$, it is clear that

$$\sup_{l \in \mathbf{Bl}_2} \sup_{n \in \mathcal{W}_N^m} \left\| \begin{matrix} k_0 n \\ M_l l + M_n n \end{matrix} \right\|^2 < mk_0^2 + 1$$

Furthermore, $\forall n \in \mathcal{W}_N^m$, $\|k_0 n\|^2 = mk_0^2$. Thus (30) is implied, as required.
**(30) $\Longrightarrow$ (31):** Assume (30) is satisfied. Then, by continuity of $D(\widetilde{\mathcal{W}}_{N,\gamma}^m, \mathcal{W}_N^m)$ at $\gamma = 0$, $\exists \gamma_0 > 0$ such that

$$\sup_{l \in \mathbf{Bl}_2} \sup_{n \in \widetilde{\mathcal{W}}_{N,\gamma_0}^m} \|M_l l + M_n n\|^2 \leq C_0 < 1$$

for some constant $C_0$. Furthermore,

$$\sup_{l \in \mathbf{Bl}_2} \sup_{n \in \widetilde{\mathcal{W}}_N^m} \|M_l l + M_n n\|^2 \leq \left(\|M_l\|_\infty + \sqrt{m}\|M_n\|_\infty\right)^2 := C_1$$

Suppose (31) is not satisfied. Then, $\forall k$, $\exists n \in \widetilde{\mathcal{W}}_N^m$ and $l \in \mathbf{Bl}_2$ such that

$$\|n\|^2 \geq m + \frac{\frac{1+C_0}{2} - \|M_l l + M_n n\|^2}{k^2}$$

In particular, let $k = \sqrt{\frac{C_1}{\gamma_0}}$. Then it follows that $n \in \widetilde{\mathcal{W}}_{N,\gamma_0}^m$. Furthermore, since $\|n\|^2 \leq m$,

$$\|M_l l + M_n n\|^2 \geq \frac{1 + C_0}{2} > C_0$$

which contradicts our assumption, as required.

□

The following corollary may be applied to controller synthesis:

**Corollary 1** *Let $G_0$ be given. For any controller $K$, let*

$$M(K) = \begin{bmatrix} M_l(K) & M_n(K) \end{bmatrix} := G_0 \star K \tag{32}$$

*Then there exists a $K$ such that*

$$\sup_{l \in \mathbf{Bl}_2, n \in \widetilde{\mathcal{W}}_N^m} \|M_l(K)l + M_n(K)n\|^2 < 1 \tag{33}$$

*if and only if there exist $K$ and $k_0$ such that $\forall k \geq k_0$,*

$$\sup_{l \in \mathbf{Bl}_2, \tilde{n} \in \mathcal{N}} \frac{1}{1 + mk^2} \left\| \begin{matrix} kV\tilde{n} \\ M_l(K)l + M_n(K)V\tilde{n} \end{matrix} \right\|^2 < 1 \tag{34}$$

The proof of the above follows from Theorem 2 and $\widetilde{\mathcal{W}}_N^m = \{n | n = V\tilde{n},\ \tilde{n} \in \mathcal{N}\}$. Next, note that constraints $\mathbf{C}_1$ to $\mathbf{C}_4$, which characterize $\mathcal{N}$, are equivalent to the following constraints:

$$\begin{array}{llll}
\mathbf{C}_1: & \|n_{i,k}\|^2 \leq 1,\ \|\tilde{n}_{i,k}\|^2 \leq 1 & 1 \leq i \leq m, & 1 \leq k \leq N \\
\mathbf{C}_2: & \|n_{i,k} + n_{j,k}\|^2 + \|\tilde{n}_{i,k}\|^2 + \|\tilde{n}_{j,k}\|^2 = \|\tilde{n}_{i,k} + \tilde{n}_{j,k}\|^2 + \|n_{i,k}\|^2 + \|n_{j,k}\|^2 & 1 \leq i < j \leq m, & 1 \leq k \leq N \\
\mathbf{C}_3: & \|n_{i,k} + \tilde{n}_{j,k}\|^2 + \|\tilde{n}_{i,k}\|^2 + \|n_{j,k}\|^2 = \|\tilde{n}_{i,k} + n_{j,k}\|^2 + \|n_{i,k}\|^2 + \|\tilde{n}_{j,k}\|^2 & 1 \leq i < j \leq m, & 1 \leq k \leq N \\
\mathbf{C}_4: & \|n_{i,k} + n_{j,k}\|^2 + \|\tilde{n}_{i,k} + \tilde{n}_{j,k}\|^2 = \|n_{i,k}\|^2 + \|n_{j,k}\|^2\| + \|\tilde{n}_{i,k}\|^2 + \|\tilde{n}_{j,k}\|^2 & 1 \leq i < j \leq m, & k = 1
\end{array}$$

Furthermore, $l \in \mathbf{Bl}_2$ is equivalent to $\|l\|^2 \leq 1$. Thus $\tilde{n} \in \mathcal{N}$ and $l \in \mathbf{Bl}_2$ can be captured in a form consistent with $\mathcal{H}$ in (11). Finally, let $\alpha = \frac{1}{\sqrt{1+mk^2}}$. Then by setting

$$G = \begin{bmatrix} 0 & k\alpha V & 0 \\ \alpha G_{11}^l & \alpha G_{11}^n V & \alpha G_{12} \\ G_{21}^l & G_{21}^n V & G_{22} \end{bmatrix} \tag{35}$$

(34) may be solved by the method presented in Section 5.1.

## 6 Computation

A controller $K$ may be found which verifies (34) using the methods of Section 5.1 for a given $k$; thus one would choose $k$, and synthesize a controller. In order to solve the original problem of (33) and (10), however, we need to ensure that $k > k_0$. It is possible to find a lower bound for $k_0$ given the open loop system $G$, $N$, and how closely we want to approximate the optimal solution. This bound, however, will more likely than not be too conservative to be of any use. In practice, $k$ should be chosen as large as the numerical algorithm allows. This is a topic for future research.

For a given $N$ and $m$, it is also worth noting how much constraints $\mathbf{C}_1$ through $\mathbf{C}_4$ cost, in terms of the number of constraints (which is linearly related to the number of decision variables required):

| | COST |
|---|---|
| $\mathbf{C}_1$ | $2mN$ |
| $\mathbf{C}_2$ | $\frac{1}{2}m(m-1)N$ |
| $\mathbf{C}_3$ | $\frac{1}{2}m(m-1)N$ |
| $\mathbf{C}_4$ | $\frac{1}{2}m(m-1)$ |
| TOTAL | $m\left(2N + (m-1)(N+\frac{1}{2})\right)$ |

Table 1: Cost of Constraints

Thus the growth is linear in $N$ and quadratic in $m$. In order to keep the computational complexity down, constraints $\mathbf{C}_2$ through $\mathbf{C}_4$ may be omitted, with the result being that each component of $n$ will tend to be a white noise signal, but may be correlated to other components. The particular nature of the problem will dictate how conservative this omission will be.

In addition, $G$ consists of an augmented version of $G_0$. In particular, $V$ results in an extra $mN$ number of states. Since the number of decision variables grows as the *square* of the number of states of the plant, quadratic growth is unavoidable, both in $N$ and in $m$.

## 7 Application to Robust $\mathcal{H}_2$ Synthesis: Robust Disturbance Rejection

We will show the types of robustness problems that may be solved using the machinery developed in this paper. Consider the setup of Figure 2. Given $P$, it is required to design $K$ such that disturbances $n_d$, along with measurement errors $n_e$, have a small effect on plant output $e$. The plant is subject to multiplicative, unstructured uncertainty $\Delta$, with associated weight $W_t$. Since $n_d$ and $n_e$ are disturbances, one may know more about

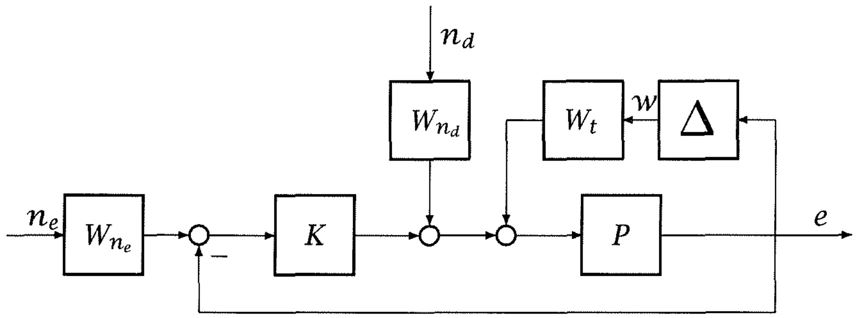


Figure 2: Robust Disturbance Rejection

their spectral content than that they are filtered, but otherwise arbitrary, $l_2$ signals. In many cases, a good model for disturbances is *colored white noise*, or filtered white noise. One may therefore want to model $n_d$ and $n_e$ as white noise signals, filtered by $W_{n_d}$ and $W_{n_e}$, respectively. An example where this may arise is $n_e$ being sensor noise and $n_d$ an impulsive (or known) disturbance (it is also straightforward to consider the case where $n_d$ is an arbitrary disturbance in $\mathbf{Bl}_2$, etc.).

The relevant equation describing the system is

$$\begin{aligned} e &= SPW_t\Delta e + SPW_{n_d}n_d + TW_{n_e}n_e \\ &= (I - SPW_t\Delta)^{-1}(SPW_{n_d}n_d + TW_{n_e}n_e) \\ &:= M_1 n_d + M_2 n_e \end{aligned} \tag{36}$$

where

$$\begin{aligned} S &:= (I + PK)^{-1} \\ T &:= PK(I + PK)^{-1} \end{aligned} \tag{37}$$

The Robust Disturbance Rejection problem is as follows:

**Robust Disturbance Rejection**

*For a given N, find an internally stabilizing K such that*

$$\sup_{\Delta \in \mathbf{B}\Delta} \sup_{(n_d, n_e) \in \mathcal{W}_N^m} \|M_1 n_d + M_2 n_e\| < 1 \tag{38}$$

The above may be converted to a condition which does not involve $\Delta$:

**Theorem 3** *K solves the Robust Disturbance Rejection problem iff K is an internally stabilizing controller and*

$$\sup_{l \in \mathbf{Bl}_2} \sup_{(n_d, n_e) \in \mathcal{W}_N^m} \|SPW_t l + SPW_{n_d}n_d + TW_{n_e}n_e\| < 1 \tag{39}$$

The proof of the above is equivalent to the one in [3], where $n_d$ and $n_e$ are assumed to be arbitrary $l_2$ disturbances (the idea is that since $\Delta$ is an arbitrary contractive map, $\Delta e$ may be replaced by $l$). The above can now be cast into the framework of Section 4. In general, many robust synthesis problems may be solved using this technique (see [3]); the only restriction is that the uncertainty (possibly more than one block) have $e$ as their input.

## 8 Example

We present a simple example for which the solution is known. In particular, we will use the machinery developed thus far to synthesize an optimal $\mathcal{H}_2$ controller for a given plant. We are not proposing this method for synthesizing optimal $\mathcal{H}_2$ controllers, since exact solutions exist, but rather wish to explore the properties of our algorithm on a simple example.

Consider the following unstable, non-minimum phase, SISO plant:

$$\hat{P} = \frac{-\lambda + \frac{1}{2}}{\lambda + \frac{2}{3}} \tag{40}$$

The goal is to minimize the $\mathcal{H}_2$ norm of the sensitivity function, $S = (1 + PK)^{-1}$. The generalized plant $G_0$ for this problem is

$$G_0 = \begin{bmatrix} 1 & -P \\ 1 & -P \end{bmatrix} \tag{41}$$

The following table summarizes the results obtained using standard synthesis methods (see [4]):

| | $\mathcal{H}_\infty$ Analysis | $\mathcal{H}_2$ Analysis | Optimal Controller |
|---|---|---|---|
| $\mathcal{H}_\infty$ Synthesis | 1.1429 | 1.1429 | $\hat{K} = \frac{5}{12}$ |
| $\mathcal{H}_2$ Synthesis | 1.7143 | 0.9897 | $\hat{K} = 1 + 0.3889\lambda$ |

Table 2: Exact synthesis results

We then repeated the synthesis for $k = 0, 1, 3, 10, 100$ with $N$ fixed at 4, and for $N = 0, 1, 2, 3, 4$ with $k$ fixed at 100. The resulting closed loop responses are depicted in Figure 3, along with the optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ results for comparison purposes.

For the first plot, the flat response is that of the optimal $\mathcal{H}_\infty$ closed loop, while the response with the highest peak is that of the optimal $\mathcal{H}_2$ closed loop. The five values in between correspond to different values of $k$, ascending values of $k$ corresponding to ascending values of the $\mathcal{H}_\infty$ norm.

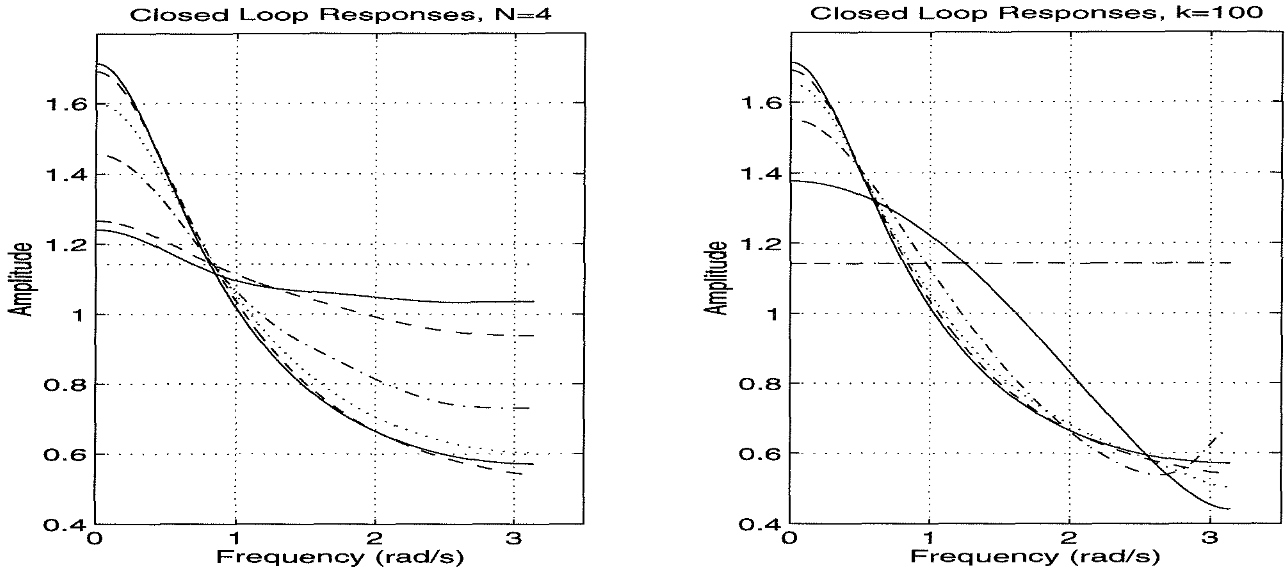


Figure 3: Synthesis with constraints

In the second plot, the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ designs are also included for comparison purposes. Ascending values of $N$ correspond to ascending values of the $\mathcal{H}_\infty$ norm. Note that only six curves seem to be present in the second plot, since the $N = 0$, $k = 100$ design is virtually identical to the $\mathcal{H}_\infty$ design, as expected.

The $N = 4$, $k = 100$ controller was

$$\hat{K} = \frac{1.0079 + 0.3953\lambda}{1 + 0.0306\lambda} \tag{42}$$

which is very close to the optimal $\mathcal{H}_2$ controller. Furthermore, the closed loop $\mathcal{H}_2$ norm for this design was 0.9899, and the $\mathcal{H}_\infty$ norm was 1.690, again extremely close to the optimal $\mathcal{H}_2$ design. Note, however, that the $N = 4$ design isn't necessarily the best one. For example, the $N = 1$ design has a closed loop $\mathcal{H}_2$ norm of 1.021 and $\mathcal{H}_\infty$ norm of 1.377; As a percentage, the $N = 1$ design has $\mathcal{H}_2$ norm which is approximately 3% larger than the $N = 4$ design, but $\mathcal{H}_\infty$ norm which is 23% smaller. The point is that most of the reduction in the $\mathcal{H}_2$ norm occurred with only one constraint; pushing harder to reduce the $\mathcal{H}_2$ norm only serves to increase the $\mathcal{H}_\infty$ norm.

There is an interesting connection between the tools developed in this paper when applied to standard $\mathcal{H}_2$ optimization and known results on $\mathcal{H}_2$ synthesis. In particular, it is known that synthesizing the optimal $\mathcal{H}_2$ controller for a given system is equivalent to synthesizing the optimal $\mathcal{H}_\infty$ controller for the same system multiplied by an appropriate weight. Thus our method can be thought of as searching for a feasible controller *and* the correct weighting function simultaneously. This interpretation is also valid for the general case, ie., when there is also a signal $l$.

## 9 Conclusions

The tools presented in this paper allow for controller synthesis when the disturbances are a combination of arbitrary, norm bounded $l_2$ signals, and signals in set $\mathcal{W}_N^m$. For a fixed $N$, the solution takes the form of an LMI, for which good numerical packages exist. To solve the problem where noise signal $n$ is perfectly white (ie., satisfies an arbitrarily large number of correlation constraints), $N$ will have to be arbitrarily large; this seems to be unavoidable, however, since the optimal controller will probably be infinite dimensional. The strength of this approach, however, is that designing for a fixed $N$ gives us both sufficient *and necessary* conditions for the existence of a controller. By inspecting the nature of $\mathcal{W}_N^m$ and the resulting closed loop system, the design engineer may then decide whether the solution is acceptable, or a larger value of $N$ is required; this is the case in the example presented, where it isn't clear whether the $N = 4$ design is better than the $N = 1$ design.

This relates to the view that optimization techniques should only be used as design tools, and should be flexible enough to allow the design engineer to customize the results to the particular problem at hand; for real problems, the objective is almost never to minimize the closed loop $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norms (or anything in between, for that matter). The best one can do is to provide flexible enough tools to automate some of the design process, and let the engineer do the rest. This is the case with standard $\mathcal{H}_\infty$ theory, where the problem of optimal control design is reduced to that of finding appropriate weights; the design becomes iterative, where weight selection is determined by the previous closed loop system. The tools provided in this paper can thus be thought of as a method to automate some of the weight selection process, and as a result, allow for more complex designs to be performed.

## Acknowledgements

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## Appendix

**Proof of Lemma 2 (see [16]):** Let

$$Q = \begin{bmatrix} \frac{1}{N+1}R(0) & \frac{1}{N}R(1) & \cdots & R(N) \\ \frac{1}{N}R(-1) & \frac{1}{N+1}R(0) & \cdots & \frac{1}{2}R(N-1) \\ \vdots & \ddots & & \vdots \\ R(-N) & \frac{1}{2}R(-N+1) & \cdots & \frac{1}{N+1}R(0) \end{bmatrix} \in \mathbb{R}^{m(N+1)\times m(N+1)} \qquad (43)$$

By Gergorshin's Circle Theorem (see [20]), we may bound each eigenvalue of $Q$ from below by

$$\frac{1-m\gamma}{N+1} - \sum_{k=1}^{N} m\gamma > \frac{1-\gamma m(N+1)^2}{N+1} \tag{44}$$

Since $\gamma < \frac{1}{m(N+1)^2}$, $Q > 0$. Thus there exists $P > 0$ such that $P^2 = Q$. Let

$$\bar{P} := \begin{bmatrix} P \\ 0 \end{bmatrix} \in \mathbb{R}^{2m(N+1)\times m(N+1)} \tag{45}$$

i.e., we have added $m(N+1)$ rows of zeros to $P$. Now define $x(t)$ as follows

$$\begin{bmatrix} x(0) \\ \vdots \\ x\left(2m(N+1)^2\right) \end{bmatrix} := \operatorname{vec}(\bar{P}) \tag{46}$$

i.e., formed by stacking the columns of $\bar{P}$ into one long vector. We claim that $R_x(\tau) = R(\tau)$ for $|\tau| \le N$. Partition $P$ as

$$P = \begin{bmatrix} P_{00} & \cdots & P_{0N} \\ \vdots & \ddots & \vdots \\ P_{N0} & \cdots & P_{NN} \end{bmatrix} \tag{47}$$

which implies

$$R(\tau) = (N+1-|\tau|) \sum_{k=0}^{N} P_{ik}P_{kj}, \quad j = i+\tau \tag{48}$$

Further partitioning each $P_{ik}$ as

$$P_{ik} = \begin{bmatrix} (P_{ik})_1 & \cdots & (P_{ik})_m \end{bmatrix} \tag{49}$$

and using the fact that $P_{kj} = P_{jk}^{\mathrm{T}}$, we have

$$R(\tau) = (N+1-|\tau|) \sum_{k=0}^{N} \sum_{l=1}^{m} (P_{ik})_l (P_{jk})_l^{\mathrm{T}}, \quad j = i+\tau \tag{50}$$

It thus follows from the definition of $x(t)$ in (46), for $0 \le \tau \le N$,

$$\begin{aligned} R_x(\tau) &= \sum_{j=0}^{N} \sum_{l=1}^{m} \sum_{i=0}^{N-\tau} (P_{ij})_l (P_{i+\tau,j})_l^{\mathrm{T}} \\ &= \sum_{i=0}^{N-\tau} \sum_{j=0}^{N} \sum_{l=1}^{m} (P_{ij})_l (P_{i+\tau,j})_l^{\mathrm{T}} \\ &= \sum_{i=0}^{N-\tau} \frac{1}{N+1-\tau} R(\tau) \\ &= R(\tau) \end{aligned} \tag{51}$$

A similar argument holds for negative $\tau$.

□

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